Exercises for the 3rd School on Belief functions
Random sets and belief functions

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1. Consider $\Omega = \{1, 2, 3\}$ with the probability measure $P(\{1\}) = 0.3, P(\{2\}) = 0.5, P(\{3\}) = 0.2$ and the random set $\Gamma : \Omega \to \mathcal{P}(\{1, 2, 3\})$ given by
$$\Gamma(1) = \{1, 2\}, \Gamma(2) = \{2, 3\}, \Gamma(3) = \{1, 2, 3\}.$$
Determine the upper and lower probabilities of the sets $A = \{1\}, B = \{1, 2\}$ and $C = \{2, 3\}$.

2. Consider the belief function Bel on $\mathcal{P}(\{1, 2, 3\})$ given by:

A	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
Bel(A)	0.1	0.2	0	0.5	0.3	0.4	1

Determine a random set having Bel as its lower probability.

3. Consider $X = \{1, 2, 3, 4\}$.
   (a) Let Π be the possibility distribution associated to the possibility distribution $\pi(1) = 0.3, \pi(2) = 0.5, \pi(3) = 1, \pi(4) = 0.7$. Determine its focal elements and its basic probability assignment.
   (b) Given the basic probability assignment $m(\{1\}) = 0.2, m(\{1, 3\}) = 0.1, m(\{1, 2, 3\}) = 0.4, m(\{1, 2, 3, 4\}) = 0.3$, determine the associated possibility measure and its possibility distribution.

4. Consider $\Omega = \{1, 2, 3\}$ with the uniform distribution, $\Gamma : \Omega \to \mathcal{P}(\{1, 2\})$, $\Gamma(\omega) = \{1, 2\}$ $\forall \omega$.
   (a) Show that $P(\Gamma) = \{(0, 1), (1/3, 2/3), (2/3, 1/3), (1, 0)\}$.
   (b) Show that $M(P^*) = \{(\alpha, 1 - \alpha) : \alpha \in [0, 1]\}$.

   Let f be the Shannon entropy of a probability measure, $f(P) = -\sum_i P(x_i) log_2(P(x_i))$.
   (c) $f(P_{U_0}) \in f(P(\Gamma)) = \{0, f(1/3, 2/3)\} = \{0, 0.92\}$.
   (d) $f(M(P_*)) = [0, 1]$.

5. Consider $\Omega = \{1, 2, 3, 4\} = X, \mathcal{A} = \mathcal{P}(\Omega), P$ the uniform distribution, and $\Gamma : \Omega \to \mathcal{P}(X)$ given by
$$\Gamma(1) = \{1\}, \Gamma(2) = \{1, 3\}, \Gamma(3) = \{2, 4\}, \Gamma(4) = \{3, 4\}.$$
Determine the Choquet integral of the mapping f given by $f(1) = 10, f(2) = 4, f(3) = 0, f(4) = 2$ with respect to P^* and P_*, and give the measurable selections that attain this lower and upper integral.

6. Consider $\Omega = \{1, 2, 3\}$ with $P(\{1\}) = 0.2, P(\{2\}) = 0.3, P(\{3\}) = 0.5$ and the random set Γ given by $\Gamma(1) = \{1, 2\}, \Gamma(2) = \{1, 3\}, \Gamma(3) = \{2, 3\}$.
   (a) Determine the lower and upper distribution functions of Γ.
   (b) Use them to find a probability measure Q such that $F_*(x) \leq F_Q(x) \leq F^*(x)$ $\forall x \in \{1, 2, 3\}$ while $Q \notin M(P_*)$.

7. Consider the belief function associated with the basic probability assignment:

A	{1}	{2}	{1,2}	{1,3}	{2,3}	{1,2,3}
m(A)	1/9	1/9	1/9	1/6	1/6	1/3

Determine the extreme points of $M(Bel)$ and its Shapley value.